Robust Coarsening in Multiscale PDEs

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1 Introduction

Consider a variationally–posed 2nd–order elliptic boundary value problem

$$a(u,v) \;\equiv\; \int_\Omega \mathcal{A}(\mathbf{x})\;\nabla u \cdot \nabla v \;=\; \int_\Omega f(\mathbf{x}) v(\mathbf{x}), \quad \text{for all} \;\; v \in H^1_0(\Omega), \qquad (1)$$

with solution $u \in H^1_0(\Omega)$ and domain $\Omega \subset \mathbb{R}^d,\ d = 2, 3$, where the coefficient tensor $\mathcal{A}(\mathbf{x})$ is highly *heterogeneous* (possibly in a spatially complicated way). We assume that $\mathcal{A}(\mathbf{x})$ is symmetric, uniformly positive definite and mildly anisotropic, i.e. $\lambda_{\min}(\mathcal{A}(\mathbf{x})) \gtrsim \lambda_{\max}(\mathcal{A}(\mathbf{x}))$ uniformly in $\mathbf{x}$. We are particularly interested in the case when the *contrast* $\max_{\mathbf{x},\mathbf{y}\in\Omega} \lambda_{\max}(\mathcal{A}(\mathbf{x}))/\lambda_{\max}(\mathcal{A}(\mathbf{y}))$ is large. Many examples of this type arise in subsurface flow modelling or in material science. The space $H^1_0(\Omega)$ is the usual Sobolev space of functions with vanishing trace on $\partial\Omega$ and $f \in H^{-1}(\Omega)$. For simplicity we assume for the remainder that $\mathcal{A}(\mathbf{x}) = \alpha(\mathbf{x}) I$, i.e. a scalar diffusion coefficient.

Let $\mathcal{T}_h$ be a simplicial triangulation of Ω and let (1) be discretised in $V_h \subset H^1_0(\Omega)$, the space of continuous, piecewise linear FE functions with respect to $\mathcal{T}_h$ that vanish on $\partial\Omega$. For simplicity let $\mathcal{T}_h$ be quasi-uniform. The a-orthogonal projection of u to V_h is denoted by u_h. In the usual nodal basis $\{\varphi_i\}_{i=1}^n$ for V_h, the problem of finding u_h reduces to the $n \times n$ linear system

$$A\mathbf{u} = \mathbf{b} \qquad (2)$$

with stiffness matrix $A = (a(\varphi_i, \varphi_j))_{i,j=1}^n$. Since the matrix A depends on α only through element averages, we can assume (w.l.o.g.) that α is piecewise constant with respect to $\mathcal{T}_h$. For simplicity we assume that α is piecewise constant with respect to some non-overlapping partitioning of Ω into open, connected Lipschitz polyhedra (polygons) $\{\mathcal{Y}_m\}_{m=1}^M$ and set $\alpha_m = \alpha|_{\mathcal{Y}_m}$.

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Especially for $d = 3$ and for problems where α varies on a small length scale $\varepsilon \ll \mathrm{diam}(\Omega)$, and thus the mesh size h needs to be very fine, multilevel iterative solvers (multigrid, domain decomposition, etc) are usually essential to solve this problem efficiently. Their scalability and robustness with respect to mesh refinement, as well as other discretisation parameters has been studied extensively. Here we will focus on their robustness with respect to coefficient variation. We will show that coefficient robustness is inherently linked to a judicious choice of coarse space V_H (related to some coarse mesh $\mathcal{T}_H$ with resolution H). If $\varepsilon \gtrsim H$ and if we can choose a coarse mesh such that all coefficient jumps are aligned with the mesh, then the coefficient robustness of standard coarse spaces has been analysed in the 90s (cf. [4, 3, 10, 16, 25, 21, 22] and the references therein). For certain methods the robustness may depend on the quasi-monotonicity of the coefficient with respect to the coarse mesh (in the sense of [3]). Substructuring-type ("exotic") coarse spaces are usually used to achieve uniform coefficient robustness. A certain amount of robustness can be recovered for standard piecewise linear coarse spaces by using the multilevel solver as a preconditioner within CG (e.g. [24]). The key tool in all these analyses is the weighted L_2–projection of Bramble and Xu [1]. It requires a piecewise constant weight with respect to the coarse mesh, an assumption that is often far too stringent in real applications. We want to move away from this and crucially here make no assumptions that the underlying coarse grids resolve the coefficients.

A lot of effort in the last 25 years has gone into the development of algebraic methods to construct coarse spaces, such as algebraic multigrid (AMG), rather than analytic/geometric ones. It has been confirmed numerically that AMG methods are in practice robust to coefficient variation when applied to (2) (i.e. the number of iterations is unaffected), and they are therefore extremely popular. However, they are built on several heuristics and so a rigorous analysis of their coefficient-robustness is difficult (see [22] for a review of existing theoretical results). Nevertheless, the key principle of these algebraic coarse spaces, namely energy minimisation [11], also underlies many other coarse spaces. To obtain rigorous coefficient–independent convergence results we will need to work in the following energy and weighted L_2-norms on $D \subset \Omega$,

$$\|v\|_{a,D} \;=\; \int_D \alpha |\nabla v|^2 \qquad \text{and} \qquad \|v\|_{0,\alpha,D} \;=\; \int_D \alpha v^2 \,,$$

respectively. When $D = \Omega$ we will usually not specify the domain explicitly.

A convenient framework to analyse most multilevel methods is the Schwarz or subspace correction framework [21, 23]. We restrict attention to the two-level overlapping additive Schwarz method and focus on the robustness of various coarse spaces for this method. We review some recent papers on the topic mainly by the author (jointly with co-workers), as well as by Efendiev et al. All the results apply immediately also to multiplicative, hybrid and non-overlapping versions of the Schwarz method (see [9, 18] for some explicit comments). Many of the results can be extended to a multilevel theory [18, 5].

2 Schwarz framework and abstract coarse spaces

Let us assume that $\{\Omega_k\}_{k=1}^K$ is an overlapping partitioning of Ω and let Ω_k° be the overlap of subdomain Ω_k, i.e. the set of points $\mathbf{x} \in \Omega_k$ that are contained in at least one other subdomain. We assume that $\mathcal{T}_h$ is aligned with this partitioning. Furthermore, let $\{\chi_k\}_{k=1}^K \subset V_h$ be an arbitrary partition of unity (POU) of FE functions subordinate to $\{\Omega_k\}_{k=1}^K$ such that $\|\chi_k\|_\infty \lesssim 1$ and $\|\nabla \chi_k\|_\infty \leq \delta_k^{-1}$, for all $k = 1, \dots, K$. Note that (due to quasi-uniformity of $\mathcal{T}_h$) we always have $\delta_k \gtrsim h$, and there is a partition of unity such that δ_k is proportional to the (minimal) width of Ω_k°. We assume as usual that each point $\mathbf{x} \in \Omega$ is contained in at most N_0 subdomains (*finite covering*).

We associate with each Ω_k the space $V_k = \{v \in V_h : \mathrm{supp}(v) \subset \overline{\Omega_k}\}$ and assume that we have an additional *coarse space*

$$V_0 = V_H = \mathrm{span}\{\Phi_j \in V_h : j = 1, \dots, N\} \subset V_h \,.$$

Let $\omega_j = \mathrm{interior}(\mathrm{supp}(\Phi_j))$ and set $H_j = \mathrm{diam}(\omega_j)$. Then $H = \max_j H_j$ is the coarse mesh size associated with V_H.

The two-level additive Schwarz preconditioner is now simply

$$M_{\mathrm{AS}}^{-1} = R_0^T A_0^{-1} R_0 \; + \; \sum\nolimits_{k=1}^K R_k^T A_k^{-1} R_k \quad \text{with} \quad A_k = R_k A R_k^T \,.$$

R_k is the matrix representation of a restriction operator from V to V_k: the simple injection operator for $k \geq 1$, and for $k = 0$ induced by the coarse space basis $\{\Phi_j\}_{j=1}^N$ so that the coarse space stiffness matrix is $A_0 = (a(\Phi_j, \Phi_\ell))_{j,\ell}^N$.

The following result can be proved in the same way as [18, Thm. 2.5]. Since it is instructive, we give an outline of the proof.

Theorem 1. *If there exists an operator $\Pi : V_h \to V_0$ such that for all $v \in V_h$*

$$\|\Pi v\|_a^2 \;\leq\; C_1 \, \|v\|_a^2 \quad \text{and} \quad \sum\nolimits_{k=1}^K \|(v - \Pi v)\nabla \chi_k\|_{0,\alpha}^2 \;\leq\; C_2 \, \|v\|_a^2 \,, \qquad (3)$$

then $\kappa(M_{\mathrm{AS}}^{-1} A) \lesssim C_1 + C_2$. The hidden constant depends on N_0.

Proof. Let $v_0 = \Pi v$ be such that (3) holds and choose $v_k = I_h(\chi_k(v - v_0))$, where I_h is the standard nodal interpolant on V_h. This interpolant is stable for all piecewise quadratic functions in the energy norm and in the weighted L_2-norm (independently of α) (cf. [18, Lem. 2.3]), and so we get

$$\begin{aligned} \sum\nolimits_{k=0}^K \|v_k\|_a^2 \;&\lesssim\; \|v_0\|_a^2 + \sum\nolimits_{k=1}^K \|\chi_k(v - v_0)\|_a^2 \\ &\lesssim\; \|v_0\|_a^2 + \sum\nolimits_{k=1}^K \|\chi_k\|_\infty^2 \|v - v_0\|_{a,\Omega_k}^2 + \|(v - v_0)\nabla\chi_k\|_{0,\alpha}^2 \,. \end{aligned}$$

Now, the boundedness of the POU functions, the finite cover assumption, as well as (3) lead to the stability estimate $\sum_{k=0}^K \|v_k\|_a^2 \lesssim (C_1 + C_2)\|v\|_a^2$. Since $v = \sum_{k=0}^K v_k$, the result follows from the abstract Schwarz theory (cf. [21]).

This result shows the importance of the choice of coarse space. Provided we have a good coarse space approximation in the weighted L_2-norm that is moreover stable in the energy norm, independently of variations in α, then the bound on the condition number for two-level additive Schwarz is also robust with respect to these variations. Note that it is crucial to use the weighted L_2 and the energy norm here to achieve coefficient-robustness, and that we only require weak L_2–approximation in regions where $\nabla\chi_k \neq 0$.

Several approaches have been studied in [2, 5, 6, 7, 8, 9, 17, 18, 19] to provide constants in (3) that are independent of α (or at least of the contrast in α) for various coarse spaces. However, in most cases the constants are not independent of $\frac{H}{\varepsilon}$, where ε is the minimal length scale at which α varies in the regions where $\nabla\chi_k \neq 0$. So unfortunately in general, to be also independent of $\frac{H}{\varepsilon}$, restrictions on the coarse mesh size are needed, at least locally.

Let us discuss the assumptions (3) a bit further. Let $\Pi v = \sum_j f_j(v)\Phi_j$, where $f_j : V_h \to \mathbb{R}$ is a suitable functional. Then

$$\|\Pi v\|_a \;=\; \big\| \sum\nolimits_j f_j(v)\Phi_j \big\|_a \;\leq\; \sum\nolimits_j |f_j(v)| \|\Phi_j\|_a \,.$$

We see that a set of coarse basis functions with bounded energy (independent of α) is beneficial. The first approaches in [8, 9, 17] attacked this target directly and aimed at bounding $\|\Phi_j\|_a$. In that case, it suffices to use the standard quasi-interpolant. Alternatively, a weighted quasi-interpolant with $f_j(v) = \int_{\omega_j}\alpha v / \int_{\omega_j}\alpha$ can be used. For certain (locally quasi-monotone) coefficients α this leads to a constant C_1 that is independent of the contrast in α, even if the energy of the basis functions is not bounded (see below).

Similar comments can be made about the second assumption in (3). Note that

$$\|(v - \Pi v)\nabla\chi_k\|_{0,\alpha}^2 \;\leq\; \begin{cases} \|\alpha|\nabla\chi_k|^2\|_\infty \|v - \Pi v\|_{0,\Omega_k^\circ}^2 \,, & \text{or} \\ \|\nabla\chi_k\|_\infty^2 \|v - \Pi v\|_{0,\alpha,\Omega_k^\circ}^2 \,. \end{cases}$$

We can either try to choose a partition of unity $\{\chi_k\}$ such that $\|\alpha|\nabla\chi_k|^2\|_\infty$ is bounded independently of α, which is again related to energy minimisation, or we can try to bound $\|v - \Pi v\|_{0,\alpha,\Omega_k^\circ}$ directly. As above, it is possible for certain (locally quasi-monotone) coefficients to achieve this and to obtain a constant C_2 that does not depend on the contrast in α (see below).

When the coefficient is not locally quasi-monotone, then it is in general necessary to enrich the coarse space, by either refining the coarse mesh locally, or by choosing more than one basis function per subdomain Ω_k, with the key tool to achieve coarse space robustness being again energy minimisation.

To highlight some of the key issues we will use a number of representative model problems shown in Figure 1. For the rest of the paper, we will only focus on cases, such as Figures 1(c-h), where it is impossible or impractical that the subdomains $\{\Omega_k\}$ and the supports $\{\omega_j\}$ of the coarse basis functions resolve the coefficient jumps. The resolved cases in Figures 1(a-b) have already been studied extensively, see e.g. [4, 3, 10, 16, 25, 21, 22, 24].

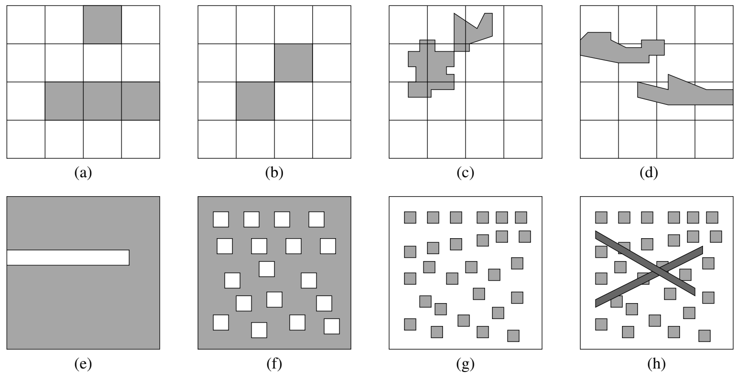

Fig. 1 Typical coefficient distributions (a) resolved; (b) not quasi-monotone; (c) neither quasi-monotone nor resolved; (d) channelised; (e) flow barriers; (f) low permeability inclusions; (g) high permeability inclusions; (h) high permeability inclusions and channels.

3 Analysis of coefficient–robustness

We present three possible approaches to try and prove coefficient robustness rigorously and thus to design robust coarse spaces. For simplicity, we assume that for each $j = 1, \ldots, N$, there exists a $k = 1, \ldots, K$ such that $\omega_j \subset \Omega_k$.

3.1 Standard quasi-interpolant & energy minimisation

The first approach makes use of the standard quasi-interpolant

$$\Pi v \;=\; \sum\nolimits_{j=1}^{N} \overline{v}_{\omega_j} \Phi_j, \qquad \text{where} \qquad \overline{v}_{\omega_j} \;=\; \tfrac{1}{|\omega_j|} \int_{\omega_j} v\,.$$

Let $\{\Phi_j\}_{j=1}^N$ be a set of bounded coarse basis functions that form a partition of unity, except in a boundary layer of width $\mathcal{O}(H)$ near $\partial\Omega$. Since each support $\omega_j \subset \Omega_k$, for some k, the supports have finite overlap. C_1 and C_2 can now be bounded independent of the contrast in α, if either

$$\gamma_2(\alpha, \{\Phi_j\}) = \max_{j=1}^{N} H_j^{2-d}\, \|\Phi_j\|_a^2 \quad \text{and} \quad \gamma_\infty(\alpha, \{\chi_k\}) = \max_{k=1}^{K} \delta_k^2\, \|\alpha^{1/2} \nabla \chi_k\|_\infty^2$$

(the so-called *coarse space and partitioning robustness indicators*) can be bounded independent of α, for some choice of the partition of unity $\{\chi_k\}_{k=1}^K$ subordinate to $\{\Omega_k\}_{k=1}^K$ (cf. [8]), or if $\gamma_\infty(\alpha, \{\Phi_j\})$ can be bounded independent of α (cf. [17]). As mentioned above, this leads to the aim to construct coarse basis functions with minimal or bounded energy. It is also at the heart of matrix-dependent prolongation operators in multigrid methods.

For certain binary coefficient distributions, e.g. for high-permeability inclusions in a low-permeability medium as depicted in Fig. 1(g), it was then possible in [8] to show (rigorously) under the assumption $\alpha \gtrsim 1$ that multiscale FEs (w.r.t. some coarse mesh $\mathcal{T}_H$) can provide such a basis $\{\Phi_j\}$, and that the indicators can be bounded independent of the contrast in α. However, they depend on H/ε, where ε is the minimum width of any island/gap.

Similarly, it was possible in [17] to show (again assuming $\alpha \gtrsim 1$) that aggregation based on a strong connection criterion (originally designed for AMG methods) leads to a coarse basis $\{\Phi_j\}$ for which the robustness indicators can be bounded independent of the contrast in α. Here the bounds depend on H/h, since the overlap between any two supports is only $\mathcal{O}(h)$.

However, this approach to analyse robustness fails even for the simpler, reverse situation of a high-permeability medium with low-permeability inclusions (e.g. Fig. 1(f)), since in this case $\gamma_2(\alpha, \{\Phi_j\})$ and $\gamma_\infty(\alpha, \{\Phi_j\})$ depend on the contrast in α for any choice of $\{\Phi_j\}$. Clearly a different quasi-interpolant Π is needed in general.

3.2 Weighted quasi-interpolant & Poincaré's inequality

The next approach to try to prove the assumptions in Theorem 1 makes use of the weighted quasi-interpolant

$$\Pi v \ = \ \sum\nolimits_{j=1}^{N} \overline{v}^{\alpha}_{\omega_j} \Phi_j, \qquad \text{where} \qquad \overline{v}^{\alpha}_{\omega_j} \ = \ \int_{\omega_j} \alpha v \Big/ \int_{\omega_j} \alpha \,.$$

We describe this approach for one of the simplest coarse spaces, the piecewise linear one. The following is taken from [18] (see also [6] for earlier results). Let V_H be the continuous, piecewise linear FE space associated with a shape-regular simplicial triangulation $\mathcal{T}_H$ of Ω, such that $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$. The functions $\{\Phi_j\}_{j=1}^N$ are the standard nodal basis for V_H. For simplicity, we assume that $\{\Omega_k\}_{k=1}^K = \{\omega_j\}_{j=1}^N$, and choose $\chi_k = \Phi_k$ (suitably modified near $\partial\Omega$), so that the assumptions on $\{\chi_k\}$ are satisfied with $\delta_k \sim H_k$.

The key observation in [18] is now that one further assumption suffices to fully describe the dependency of the constants C_1 and C_2 in (3) on α:

Assumption 1. Let $\omega_T = \bigcup_{\{k:\omega_k \cap T \neq \emptyset\}} \omega_k$ and $H_T = \mathrm{diam}(\omega_T)$, for $T \in \mathcal{T}_H$, and assume that there exists a $C_T^* > 0$ such that, for all $v \in V_h$, either

$$\inf_{c\in\mathbb{R}} \int_{\omega_T} \alpha (v-c)^2 \, d\mathbf{x} \ \lesssim \ C_T^* H_T^2 \int_{\omega_T} \alpha |\nabla v|^2 \, d\mathbf{x}, \qquad \text{or} \tag{4}$$

$$\partial\omega_T \cap \partial\Omega \neq \emptyset \quad \text{and} \quad \int_{\omega_T} \alpha v^2 \, d\mathbf{x} \ \lesssim \ C_T^* H_T^2 \int_{\omega_T} \alpha |\nabla v|^2 \, d\mathbf{x} \,. \tag{5}$$

Proposition 1. *Let Assumption 1 hold. Then* $C_1 + C_2 \ \lesssim \ C^* = \max_{T\in\mathcal{T}_H} C_T^* \,.$

Proof. Let $v \in V_h$ and $v_0 = \sum_{j=1}^{N} \overline{v}^{\alpha}_{\omega_j} \varPhi_j$. By the Cauchy-Schwarz inequality we have $|\overline{v}^{\alpha}_{\omega_j}|^2 \leq \int_{\omega_j} \alpha v^2 / \int_{\omega_j} \alpha$, and so, using the fact that $\varPhi_j \leq 1$,

$$\int_T \alpha v_0^2 \;\leq\; \sum_{j:\omega_j \cap T \neq \emptyset} \frac{\int_{\omega_j} \alpha v^2}{\int_{\omega_j} \alpha} \int_T \alpha \varPhi_j^2 \;\leq\; \int_{\omega_T} \alpha v^2 \,,$$

which also implies $\int_T \alpha (v - v_0)^2 \lesssim \int_{\omega_T} \alpha v^2$. Now, multiplying the left hand side by $|\nabla \chi_k|_T^2$ (which is a constant $\sim H_T^{-2}$) and summing over $k \geq 1$, we get

$$\sum_{k=1}^{K} \|(v - v_0)\nabla \chi_k\|^2_{0,\alpha,T} \;\lesssim\; H_T^{-2} \int_{\omega_T} \alpha v^2 \,. \tag{6}$$

If $\{\varPhi_j\}$ forms a partition of unity on all of ω_T (i.e. if $\partial\omega_T \cap \partial\varOmega = \emptyset$), we can replace v in (6) by $\hat{v} = v - c$, for any $c \in \mathbb{R}$, without changing the integral on the left hand side. Otherwise we set $\hat{v} = v$. In both cases, by Assumption 1

$$\int_{\omega_T} \alpha \hat{v}^2 \;\lesssim\; C_T^* \, H_T^2 \int_{\omega_T} \alpha |\nabla v|^2 \,. \tag{7}$$

Combining (6) and (7) and summing over all $T \in \mathcal{T}_H$ gives the bound for C_2.
The bound for C_1 can be established in a similar way (cf. [18, Lem. 4.1]).

Assumption 1 postulates the existence of a discrete weighted Poincaré/ Friedrichs–type inequality on each ω_T. It always holds, but in general the constants C_T^* will not be independent of $\alpha|_{\omega_T}$ and H_T/h. As described in detail in [18, §3] (see also [14, 15, 13]), to obtain independence of α, we require a certain local quasi–monotonicity of α on each of the regions ω_T.

Weighted Poincaré Inequalities. Let us consider a generic coarse element $T \in \mathcal{T}_H$ and define the following subsets of ω_T where α is constant:

$$\omega^m = \omega_T \cap \mathcal{Y}_m, \qquad m = 1, \ldots, M.$$

By $\mathcal{I}_T \subset \{1, \ldots, M\}$ we denote the index set of all regions ω^m that are non-empty. Let us assume w.l.o.g. that each of these subregions is connected. We generalise now the notion of quasi-monotonicity coined in [3] by considering the following three (two) directed combinatorial graphs $\mathcal{G}^{(k)} = (\mathcal{N}, \mathcal{E}^{(k)})$, $0 \leq k \leq d-1$, where $\mathcal{N} = \{\omega^m : m \in \mathcal{I}_T\}$ and the edges are ordered pairs of vertices. We distinguish between three (two) different types of connections.

Definition 1. Suppose that $\gamma^{m,m_2} = \overline{\omega}^m \cap \overline{\omega}^{m_2}$ is a non-empty manifold of dimension k, for $0 \leq k \leq d-1$. The ordered pair (ω^m, ω^{m_2}) is an edge in $\mathcal{E}^{(k)}$, if and only if $\alpha_m \lesssim \alpha_{m_2}$. The edges in $\mathcal{E}^{(k)}$ are said to be of *type-k*.

In addition, for $1 \leq k \leq d-1$, we assume that

- $\mathrm{meas}(\gamma^{m,m_2}) \sim \mathrm{meas}(\omega^m \cup \omega^{m_2})^{k/d}$, and
- γ^{m,m_2} is sufficiently regular, i.e. it is a finite union of shape–regular k-dimensional simplices of diameter $\sim \mathrm{meas}(\gamma^{m,m_2})^{1/k}$.

Quasi-monotonicity is related to the connectivity in $\mathcal{G}^{(k)}$. Let $m_* \in \mathcal{I}_T$ be the index of the region ω^{m_*} with the largest coefficient: $\alpha_{m_*} = \max_{m \in \mathcal{I}_T} \alpha_m$.

Definition 2. The coefficient α is *type-k quasi-monotone on* ω_T, if there is a path in $\mathcal{G}^{(k)}$ from any vertex ω^m to ω^{m_*}.

The following lemma summarises the results in [14, 15, 13]. The existence of a benign constant C_T^* that is independent of α is directly linked to quasi-monotonicity, the way in which C_T^* depends on H_T/h to the type.

Lemma 1. *Let $\omega_T \subset \mathbb{R}^d$, $d = 2, 3$. If α is type-k quasi-monotone on ω_T, then (4) holds with*

$$C_T^* \;=\; \begin{cases} 1, & \textit{if} \;\; k = d-1, \\ 1 + \log\left(\frac{H_T}{h}\right), & \textit{if} \;\; k = d-2, \\ \frac{H_T}{h}, & \textit{if} \;\; k = d-3. \end{cases} \tag{8}$$

A similar result can also be established in the case where $\partial\omega_K \cap \partial\Omega \neq \emptyset$, i.e. the case of Friedrichs inequality (5), see e.g. [18, §3] for details.

Quasi–monotonicity is crucial. If the coefficient is not quasi-monotone, e.g. the situation in Figure 1(d), then C^* cannot be bounded independent of α. See [18, Ex. 3.1] for a counter example. If the coarse mesh is not adjusted in certain critical areas of Ω, then V_H is in general not robust. The numerical results in [18] show that this is indeed the case and that quasi–monotonicity is necessary and sufficient. However, a few simple adjustments suffice, namely $\mathcal{T}_H$ has to be sufficiently fine in certain "critical" areas of Ω:

1. Choose $H_T \le \varepsilon_m$, for all $T \in \mathcal{T}_H$ that intersect a region $\mathcal{Y}_m$ that is bordered by two regions $\mathcal{Y}_{m'}$ and $\mathcal{Y}_{m''}$ with $\alpha_{m'} \gg \alpha_m$ and $\alpha_{m''} \gg \alpha_m$. Here ε_m denotes the width of $\mathcal{Y}_m$ at its narrowest point. This ensures that α is quasi-monotone on all regions ω_T that intersect $\mathcal{Y}_m$.
2. Choose $H_T \lesssim h$, near any point or edge where α is only type-$(d-2)$ or type-$(d-3)$ quasi–monotone, i.e. near any cross point.

Usually a logarithmic growth $C^* \sim \max_T \log(H_T/h)$ is acceptable, and so even regions where the coefficient is type-$(d-2)$ quasi-monotone do not require any particular attention.

For an arbitrary piecewise constant coefficient function α there will often only be a relatively small (fixed) number of regions ω_T where α is not quasi-monotone (see e.g. Figures 1(b-e)). Therefore it is very easy to ensure through some local refinement of $\mathcal{T}_H$ near these regions that $C^* \sim 1$ (or $C^* \sim \log(H/h)$). Note that crucially, this local refinement does not mean that $\mathcal{T}_H$ has to be aligned with coefficient jumps anywhere in Ω. The coarse grid merely has to be sufficiently fine in regions where α is not quasi-monotone. Ideas on how to adapt $\mathcal{T}_H$ in such a way are suggested in [18].

"Exotic" coarse spaces. Substructuring–type ("exotic") coarse spaces (as suggested in [4, 3, 16]) can be analysed in a similar way. Here the coarse basis functions are constructed as a-harmonic extensions of face, edge or vertex "cut" functions associated with a non-overlapping decomposition $\mathcal{T}_H$ of the domain. This decomposition may be related to the overlapping partitioning $\{\Omega_k\}$, or it may come from a separate coarse grid (not necessarily simplicial). If the coefficient does not vary along any of the edges/faces of $\mathcal{T}_H$, then the space can be analysed like the piecewise linear one above, using in addition the energy minimising property of the a-harmonic extension (cf. [14]). If the coefficient does vary along an edge/face, then special weighted Poincaré inequalities for functions with vanishing weighted averages across edges/faces are required. These have recently been introduced in the context of FETI-DP methods in [12], which also analyses the robustness of the "cut" functions. An explicit analysis in the context of overlapping Schwarz does not yet exist.

3.3 Abstract minimisation with functional constraints

An alternative to refining the coarse mesh in regions where α is not type–$(d-1)$ or type–$(d-2)$ quasi-monotone, is to associate more than one basis function (with possibly identical supports) with each subdomain Ω_k. Let

$$V_0 = \mathrm{span}\{\Phi_{k,j} = I_h\left(\chi_k \Psi_{k,j}\right) : j = 1, \ldots, N_k, \quad k = 1, \ldots, K\},$$

where $\Psi_{k,j}$, $j = 1, \ldots, N_k$, are suitable FE functions in $V_h(\overline{\Omega}_k)$ (that do not vanish on $\partial\Omega_k$) such that the functions $\{\Phi_{k,j}\} \subset V_h$ are linearly independent. Good choices for the functions $\Psi_{k,j}$ are the lowest modes of local eigenproblems, or more generally, energy minimising functions that satisfy suitable constraints. The following analysis is from [19] (see [7, 2] for related work).

In particular, let us assume that, for every Ω_k, we have a collection of linear functionals $\{f_{k,j}\}_{j=1}^{N_k} \subset V_h(\overline{\Omega}_k)'$ and let

$$\Psi_{k,j} = \arg \min_{v \in V_h(\overline{\Omega}_k)} |v|_a^2, \quad \text{subject to} \quad f_{k,l}(\Psi_{k,j}) = \delta_{jl} \quad j, l = 1, \ldots, N_k \,. \tag{9}$$

Now, for any $v \in V_h$, choose the following quasi-interpolant

$$\Pi v = \sum\nolimits_{k=1}^{K} I_h\left(\chi_k \Pi_{\Omega_k} v\right) , \quad \text{where} \quad \Pi_{\Omega_k} v = \sum\nolimits_{j=1}^{N_k} f_{k,j}(v|_{\Omega_k}) \Psi_{k,j} \,,$$

i.e. a linear combination of the basis functions $\Phi_{k,j}$ with weights $f_{k,j}(v|_{\Omega_k})$. Then the bounds on C_1 and C_2 in Theorem 3 depend only on the stability and on the local L_2-approximation properties of Π_{Ω_k} on each Ω_k.

Theorem 2. *For all $k = 1, \ldots, K$ and for all $v \in V_h(\overline{\Omega}_k)$, let*

$$\|\Pi_{\Omega_k} v\|_{a,\Omega_k}^2 \le \|v\|_{a,\Omega_k}^2 \quad \textit{and} \quad \|v - \Pi_{\Omega_k} v\|_{0,\alpha,\Omega_k}^2 \lesssim \mathrm{diam}(\Omega_k)^2 \|u\|_{a,\Omega_k}^2 \,. \tag{10}$$

Then $C_1 = \mathcal{O}(1)$ and $C_2 \lesssim (\mathrm{diam}(\Omega_k)/\delta_k)^2$.

Proof. See [19, Thm. 5.1].

Note that the minimisation problems in (9) are local to each subdomain. There are suitable choices for the functionals $f_{k,j}$ that guarantee (10) and that lead to practical algorithms to construct the functions $\Psi_{k,j}$, $j = 1, \dots, N_k$:

- $f_{k,j}(v) = (\Psi_{k,j}, v)_{0,\alpha,\Omega_k}$ where $\Psi_{k,j}$ is the jth eigenfunction corresponding to the variational eigenproblem: Find $\eta \in V_h(\overline{\Omega}_k)$ and $\lambda \geq 0$, such that

$$a(\eta, w) = \lambda(\eta, w)_{0,\alpha,\Omega_k}, \quad \text{for all} \quad w \in V_h(\overline{\Omega}_k). \tag{11}$$

  This has first been suggested and analysed in [7].
- $f_{k,j}(v) = (\Psi_{k,j}, v)_{0,\alpha,\partial\Omega_k}$ where $\Psi_{k,j}$ is the jth eigenfunction corresponding to a variational eigenproblem similar to (11), but with $(\eta, w)_{0,\alpha,\partial\Omega_k}$ instead of $(\eta, w)_{0,\alpha,\Omega_k}$ on the right hand side of (11), i.e. an eigenproblem of Steklov-Poincaré type. This has been analysed in [2].
- $f_{k,j}(v) = \overline{v}^{\alpha}_{D_{k,j}}$ where $\{D_{k,j}\}_{j=1}^{N_k}$ is a suitable non-overlapping partitioning of Ω_k such that the weighted Poincaré inequality (4) holds on each $D_{k,j}$ (e.g. $D_{k,j} = \Omega_k \cap \mathcal{Y}_j$). The construction of $\{\Psi_{k,j}\}$ requires the solution of N_k local saddle point systems and was suggested and analysed in [19].

It has been shown in [7, 2] how (10) can be proved (directly) in the first two cases, essentially based on the observation that the coarse space consists of the lowest modes corresponding to the operator pencil associated to the energy and to the weighted L_2-norm. But the assumptions can be proved for a much wider class of functionals using the following abstract approximation result in [19]. This result is related to the classical Bramble-Hilbert lemma.

Abstract Approximation Result. Consider an abstract symmetric and continuous bilinear form $a(\cdot,\cdot) : V \times V \mapsto \mathbb{R}$, as well as a collection of linear functionals $\{f_l\}_{l=1}^m \subset V'$, where $V \subset \mathcal{H}$ and $\mathcal{H}$ is a Hilbert space with norm $\|\cdot\|$. We make the following assumptions on $a(\cdot,\cdot)$, V, $\mathcal{H}$, $\|\cdot\|$ and $\{f_l\}$:

A1. $a(\cdot,\cdot)$ is positive semi-definite and defines a semi-norm $|\cdot|_a$ on V, i.e.

$$|v|_a^2 = a(v, v) \geq 0, \quad \text{for all} \quad v \in V.$$

In addition, for $v \in V$, the expression $\sqrt{\|v\|^2 + |v|_a^2}$ defines a norm on V.

A2. Let c_q be a generic constant. For all $\mathbf{q} \in \mathbb{R}^m$ there exists a $v_{\mathbf{q}} \in V$ with

$$f_l(v_{\mathbf{q}}) = q_l, \quad \text{and} \quad \|v_{\mathbf{q}}\| \lesssim c_q \|\mathbf{q}\|_{l^2(\mathbb{R}^m)}.$$

A3. There are two constants c_a and c_f such that

$$\|v\|^2 \leq c_a |v|_a^2 + c_f \sum\nolimits_{l=1}^m |f_l(v)|^2\,, \quad \text{for all} \;\; v \in V. \tag{12}$$

Now, as in the specific case above, define for all $v \in V$,

$$\pi v \;=\; \sum_{l=1}^m f_l(v)\psi_l\,, \quad \text{where} \quad \psi_l = \arg\min_{v \in V} |v|_a^2, \quad \text{subject to} \quad f_l(\psi_j) = \delta_{jl}\,.$$

Then the following inequalities hold; see [19, Thm. 3.3].

Theorem 3. *Let Assumptions* **A1-3** *be satisfied. Then, for all* $u \in V$*:*

$$|\pi u|_a \leq |u|_a \qquad \textit{and} \qquad \|u - \pi u\| \leq \sqrt{c_a}|u|_a \,. \tag{13}$$

(Note that they are independent of the constants c_q *and* c_f *in* **A2** *and* **A3***.)*

In the specific case considered above, on an arbitrary subdomain Ω_k, Assumption **A1** is naturally satisfied with $\mathcal{H} = L_2(\Omega_k)$ and $\|\cdot\| = \|\cdot\|_{0,\alpha,\Omega_k}$. Assumption **A2** merely ensures that the linear functionals are linearly independent. Thus, the question of coarse space robustness is reduced to verifying Assumption **A3**. For one functional, i.e. for $m = 1$, this reduces to the weighted Poincaré inequality in Section 3.2 and to the restrictions on the coefficients made there. For more than one functional, it opens the possibility to get coefficient robustness even in the case of non-quasi-monotone coefficients, such as those depicted in Figures 1(b-d) and even (h). See [19, 7, 2] for the complete analysis and some numerical experiments that confirm the robustness for the functionals defined on the previous page. See also [20] for a more recent extension to systems of elliptic PDEs (such as linear elasticity).

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